Elementary Homotopy Theory I

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1 Elementary Homotopy Theory

Definition 1 *Two maps $f, g : X \to Y$ between spaces X, Y are said to be* ***homotopic*** *if there exists a continuous map $H : X \times I \to Y$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. The map H is said to be a* ***homotopy*** *from f to g, and we write $H : f \simeq g$.* $\square$

We will often write $(x, t) \mapsto H_t(x) = H(x, t)$ for the action of a homotopy. This notation motivates us to consider some other notions of homotopy. The reader who favours a more hands-on approach may prefer to return to these paragraphs after studying the examples in 1.1.

Definition 2 *For spaces X, Y we write $Top(X, Y)$ for the set of continuous maps $X \to Y$, and write $C(X, Y)$ for the space of such maps given the compact-open topology. In the special case that $X = I$, we write $C(I, Y) = Y^I$.* $\square$

We now have a correspondence between (not necessarily continuous) functions $X \times Y \to Z$, $X \to C(Y, Z)$, and $Y \to C(X, Z)$. In general we have the following relation between these maps.

Lemma 1.1 *For spaces X, Y, Z the function*

$$(-)^\# : Top(X \times Y, Z) \to Top(X, C(Y, Z)), \qquad f \mapsto [f^\# : x \mapsto [y \mapsto f(x, y)]] \tag{1.1}$$

is an injection of sets. If the evaluation map $ev_{Y,Z} : C(Y, Z) \times Y \to Z$, $(g, y) \mapsto g(y)$, is continuous, then $(-)^\#$ is a bijection of sets and

$$(-)^\flat : Top(X, C(Y, Z)) \to Top(X \times Y, Z), \qquad g \mapsto [g^\flat : (x, y) \mapsto g(x)(y)] \tag{1.2}$$

is its set-theoretic inverse. In particular these functions are inverse bijections when Y is locally compact[1]*.* ∎

[1] We say a space X is **locally compact at** a point $x \in X$ if each neighbourhood of x contains a compact neighbourhood this point. We say that X is **locally compact** if it is locally compact at each of its points. Warning: different definitions of local compactness appear in the literature.

Now, I is locally compact, so there is a one-to-one correspondence between continuous maps $X \times I \to Y$ and continuous maps $X \to C(I, Y) = Y^I$. Thus we could reformulate Definition 1 by saying that a *homotopy* $G : f \simeq g$ is a map

$$G : X \to Y^I \tag{1.3}$$

such that $G(x)(0) = f(x)$ and $G(x)(1) = g(x)$ for all $x \in X$. According to Lemma 1.1 this notion is entirely equivalent to that introduced previously, and in the sequel we shall make use of both points of view, since each has its own merits.

Finally there is a third way to think of a homotopy, which is as a path $I \to C(X, Y)$. The problem with this is that it only works well when our spaces are sufficiently nice.

Lemma 1.2 *If $f \simeq g : X \to Y$, then these maps are connected by a path $I \to C(X, Y)$. If X is locally compact, then the converse is true, and $f \simeq g$ if and only these maps are connected by a path $I \to C(X, Y)$.* ∎

This would be our preferred way to work with homotopies were it not for the obvious flaw. Unfortunately it is unavoidable that problematic spaces will be introduced at some point[2]. We will return to this idea at a later time.

We now have three ways to think of a homotopy. However we choose to do so we will display it diagrammatically as

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & Y. \end{tikzcd} \tag{1.4}$$

We think of the spaces X, Y as 0-dimensional points, the maps $f, g : X \to Y$ as 1-dimensional lines, and the homotopy $H : f \simeq g$ as a 2-dimensional area. We'll see later that it becomes useful to extend these diagrams to higher dimensions, although we will not push this far.

Now, saying that f relates to g if and only if $f \simeq g$ gives a relation on $Top(X, Y)$. If X is locally compact, then this is the same as the relation of paths on $C(X, Y)$ (cf. Le.1.2), but in general this will not be so. In any case, the following observation is fundamental.

Proposition 1.3 *The relation of homotopy on maps $X \to Y$ is an equivalence relation and a congruence with respect to composition.* ∎

The proof is immediate and left as an exercise. The reader will, however, find the majority of it in contained within the following remarks, which we include to set notation and to explain our use of the term *congruence.*

1. For any map $f : X \to Y$ there is the **constant**, or trivial, homotopy $f \simeq f$ defined by $f \circ pr_X : X \times I \to X \to Y$. It will sometimes be convenient to denote this homotopy simply by f when no confusion will arise.

[2]For example $\mathbb{R}$ is locally compact, but its subspace $\mathbb{Q}$ is not, so local compactness is not hereditary. If Q denotes the rationals with the discrete topology, then it is locally compact, but the image of the continuous surjection $Q \to \mathbb{Q}$ is not locally compact. In general local compactness is neither a hereditary or divisible topological property.

2. If $H : f \simeq g : X \to Y$ is given, then $-H$ denotes the homotopy $g \simeq f$ defined by
$$(-H)(x,t) = H(x, 1-t). \tag{1.5}$$

3. If $G : f \simeq g$ and $H : g \simeq h$ are homotopies of maps $X \to Y$, then $G + H$ is the homotopy $f \simeq h$ defined by
$$(G+H)(x,t) = \begin{cases} G(x,2t) & 0 \le t \le \frac{1}{2} \\ H(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \qquad x \in X,\ t \in I. \tag{1.6}$$
We call $G + H$ the **vertical composition** of the two homotopies. For obvious reasons we will also call it **addition** of homotopies.

4. The statement that homotopy is a congruence means the following: if $f \simeq g : X \to Y$, then $hfk \simeq hgk$ whenever $k : W \to X$ and $h : Y \to Z$ are given. This gives rise to a **horizontal composition** of homotopies. Namely if $H : f \simeq g : X \to Y$ and $K : h \simeq l : Y \to Z$ are given, then we define the horizontal composition $K \star H : hf \simeq lg$ by
$$(K \star H)(x,t) = K(H(x,t),t) = K_t H_t(x), \qquad x \in X,\ t \in I. \tag{1.7}$$

The words *vertical* and *horizontal* come from the diagramatic description. The vertical composition described above is suggested by the following pasting scheme

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow G"] \arrow[r, phantom, shift right=3, "\Downarrow H"] & Y \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, "\Downarrow G+H"] & Y \end{tikzcd} \tag{1.8}$$

while the horizontal composition is suggested by

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow H"] & Y \arrow[r, bend left=50, "h"] \arrow[r, bend right=50, "l"'] \arrow[r, phantom, "\Downarrow K"] & Z \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left=50, "hf"] \arrow[r, bend right=50, "lg"'] \arrow[r, phantom, "\Downarrow K\star H"] & Z. \end{tikzcd} \tag{1.9}$$

The two methods of composition are related by the following useful fact, whose proof is a direct check left to the reader.

Proposition 1.4 (Interchange Law) *Let X, Y, Z be spaces and let $f_0, f_1, f_2 : X \to Y$ and $g_1, g_2, g_2 : Y \to Z$ be maps. Assume given homotopies $G : f_0 \simeq f_1$, $H : f_1 \simeq f_2$, $K : g_0 \simeq g_1$ and $L : g_1 \simeq g_2$. Then the following two homotopies $g_0 f_0 \simeq g_2 f_2$ coincide*
$$(K+L) \star (G+H) = (K \star G) + (L \star H). \quad \blacksquare \tag{1.10}$$

Notice the simple form the statement takes when expressed diagrammatically

$$\begin{tikzcd} X \arrow[r, bend left=50, "f_0"] \arrow[r, "f_1" description] \arrow[r, bend right=50, "f_2"'] \arrow[r, phantom, shift left=3ex, "\Downarrow G"] \arrow[r, phantom, shift right=3ex, "\Downarrow H"] & Y \arrow[r, bend left=50, "g_1"] \arrow[r, "g_0" description] \arrow[r, bend right=50, "g_2"'] \arrow[r, phantom, shift left=3ex, "\Downarrow K"] \arrow[r, phantom, shift right=3ex, "\Downarrow L"] & Z \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[rr, bend left=30, "g_0 f_0"] \arrow[rr, "g_1 f_1" description] \arrow[rr, bend right=30, "g_2 f_2"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow K \star G"] \arrow[rr, phantom, shift right=3ex, "\Downarrow L \star H"] & & Z. \end{tikzcd} \tag{1.11}$$

Next, having discussed some necessary ideas, let us begin to contemplate some examples. Although the examples we give here are simple, their purpose is to serve as a very intuitive introduction to the theory.

Example 1.1

1. If V is a real vector space, then any two maps $f, g : X \to V$ are homotopic, say, by a *linear* homotopy
$$F(x,t) = (1-t)f(x) + tg(x) = f(x) + t(g(x) - f(x)). \tag{1.12}$$
2. More generally, if $C \subseteq V$ is any convex subset, then any two maps $f, g : X \to C$ are homotopic.
3. More generally still, any two maps $f, g : X \to C$ into a subset $C \subseteq V$ which is $*$-convex around a point $v_0 \in C$ (i.e. if $v \in C$ is any point, then all points on the line segment $(1-t)v + tv_0$, $t \in I$, lie in C) are homotopic. Here we need to use the transitivity of the homotopy relation.
4. If A is an affine space, then any two maps $f, g : X \to A$ are homotopic, since a choice of any point turns A into a vector space. In this case we do not have a canonical homotopy.
5. Reversing the situation, let $C \subseteq V$ be a convex subset. Then any pair of maps $C \to X$ into a space X are homotopic. To see this it will be sufficient to show that any map $f : C \to X$ is homotopic to a constant map, and for this we simply use the homotopy
$$F_t(v) = f\left((1-t)v + tv_0\right), \qquad v \in C,\ t \in I. \tag{1.13}$$
where $v_0 \in C$ is some chosen point. $\square$

These examples are important enough to motivate a definition to capture their behaviour. First some terminology. We call a map $f : X \to Y$ **inessential** if it is homotopic to the constant map at some point $y_0 \in Y$. A map is **essential** if it is not inessential. Most authors also say that an inessential map is **null-homotopic**, and this is terminology which we will also adopt.

Definition 3 *A space X is said to be **contractible** if the identity id_X is inessential.* $\square$

Thus id_X is homotopic to the constant map at some point $x_0 \in X$. In particular X is nonempty. We call a choice of null-homotopy $F : id_X \simeq x_0$ a **contraction**, or **contracting homotopy**.

Example 1.2

1. A real vector space V is canonically contractible, since the two maps $id_V, v_0 : V \to V$ are linearly homotopic.

2. An affine space A is contractible, but not canonically so.

3. A nonempty indiscrete space X is contractible:

$$(x,t) \mapsto \begin{cases} x & t < 1 \\ x_0 & t = 1. \end{cases} \tag{1.14}$$

4. The two-point space $\mathbb{S} = \{0,1\}$ with topology $\{\emptyset, \{1\}, \mathbb{S}\}$ is called the **Sierpinski space**. Although it is not indiscrete, it is still contractible. □

Here is a pair of simple but useful observations.

Proposition 1.5 *A nonempty space Y is contractible if and only if for any space X, any pair of maps $f, g : X \to Y$ are homotopic.*

Proof If Y is contractible and F_t contracts it to a point y_0, then $F_t f$ and $F_t g$ are homotopies $f \simeq y_0$ and $g \simeq y_0$. In the other direction we can take $X = Y$, $f = id_Y$, $g = y_0$. ∎

Corollary 1.6 *A contractible space is path connected and contracts to any of its points.* ∎

It may seem surprising, but any space X embeds into a contractible space. Define the (unreduced) **cone** on X to be the quotient space

$$\widetilde{C}X = \frac{X \times I}{X \times \{1\}}, \tag{1.15}$$

and let $j_X : X \hookrightarrow \widetilde{C}X$ be the map $x \mapsto [x, 0]$, where we denote the equivalence classes in $\widetilde{C}X$ with square brackets. By dragging the cone up to its point we get a canonical contraction

$$F_s[x,t] = [x, (1-s)t + s], \qquad [x,t] \in \widetilde{C}X, s \in I. \tag{1.16}$$

Moreover, the map j_X is a closed embedding, since for any $0 < \epsilon < 1$ the image of $X \times [0, \epsilon)$ in $\widetilde{C}X$ is a neighbourhood of $j_X(X)$ which carries the product topology.

A consequence of this construction which we will look to lever in future is the following.

Proposition 1.7 *A map $f : X \to Y$ is null-homotopic if and only if it extends over $\widetilde{C}X$.*

Proof If $H : f \simeq y_0$ is a null-homotopy, then $H_1(X) \subseteq \{y_0\}$, so $H : X \times I \to Y$ factors over the quotient to give a map $\widetilde{f} : \widetilde{C}X \to Y$ satisfying $\widetilde{f} j_X = H_0 = f$. On the other hand, if f factors through the contractible space $\widetilde{C}X$, then, according to Propositions 1.3 and 1.5, it is null-homotopic. ∎

Note that the cone construction is functorial. If $f : X \to Y$ is a map, then the canonical null-homotopy of $j_Y f$ induces a continuous map $\widetilde{C}f : \widetilde{C}X \to \widetilde{C}Y$ making

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j_X"'] & Y \arrow[d, "j_Y"] \\ \widetilde{C}X \arrow[r, "\widetilde{C}f"] & \widetilde{C}Y \end{tikzcd} \tag{1.17}$$

commute strictly.

According to Proposition 1.5 and its corollary, through the eyes of a homotopy theorist, a space which is contractible is no more interesting than a point. Really we should like a more formal way to be able to compare spaces up to homotopy.

Definition 4 *A map $f : X \to Y$ is said to be a* ***homotopy equivalence*** *if it is invertible up to homotopy. That is, if there exists a map $g : Y \to X$ and a pair of homotopies $gf \simeq id_X$ and $fg \simeq id_Y$.* $\square$

The definition is symmetric: if $f : X \xrightarrow{\simeq} Y$ as in the definition is a homotopy equivalence, then so is its **homotopy inverse** $g : Y \xrightarrow{\simeq} X$. We say that spaces X, Y are **homotopy equivalent**, and write $X \simeq Y$, when there is a homotopy equivalence between them (pointing in either direction). The relation on spaces given by 'homotopy equivalent to' is an equivalence relation, and its equivalence classes are called **homotopy types**.

Example 1.3

1. A homeomorphism $X \cong Y$ is a homotopy equivalence. In particular homotopy equivalence is a coarser relation than homeomorpism. Thus if $X \not\simeq Y$, then $X \ncong Y$. This observation is useful because it is generally a much easier task to show that two spaces are not homotopy equivalent than it is to show that they are not homeomorphic. For instance, the homotopy and cohomology groups which we will define can be used to obstruct the presence of a homotopy equivalence. In turn a lack of homotopy equivalence is an obstruction to the presence of a homeomorphism.

2. If $f : X \xrightarrow{\simeq} Y$ is a homotopy equivalence and $f \simeq g$, then g is a homotopy equivalence.

3. If X is contractible, then the inclusion $x_0 \hookrightarrow X$ of any of its points is a homotopy equivalence whose inverse is the collapse map $X \to x_0$.

4. There is a famous homotopy equivalence

$$\text{doughnut} \simeq \text{coffee cup}. \tag{1.18}$$

   Attempt to visualise this. Both objects have one hole, and the placement of this hole and the 'dent' in the coffee cup are really inconsequential to the deformation we must perform.

5. There are, up to homeomorphism, three possible topologies on a set with two points $S = \{x, y\}$. How many distinct homotopy types are there amongst these spaces? (Solution: The Sierpinski space and the indiscrete space are contractible. S^0 is not even path-connected.)

6. Let $T = \{x, y, z\}$ be a set with three distinct points. There are twenty nine possible topologies on this set in total, and nine of them are inequivalent. Three of these inequivalent topologies are disjoint unions of contractible spaces, and thus are not connected. The six remaining topologies are connected. How many distinct homotopy types are there amongst these spaces? (Solution: The answer is, of course, three. But how many of spaces are there in each type?).

Example 1.4 Topological properties are generally not invariant under homotopy equivalence. Things like the Hausdorff, regularity, and normality separation conditions can be completely destroyed. For instance

$$\mathbb{R}^n \simeq * \simeq \mathbb{S} \tag{1.19}$$

and the Sierpinski space $\mathbb{S}$ is not even T_1.

On the other hand properties like connectedness and path-connectedness *are* invariants of homotopy type. The topologists's sine curve is the connected but not path-connected subspace of $\mathbb{R}^2$ given by $\mathcal{S} = \{(x, \sin(1/x) \in \mathbb{R}^2 \mid x \in (0,1]\} \cup (\{0\} \times [-1,1])$. No two of the three spaces I, S^0 and $\mathcal{S}$ can be homotopy equivalent because of the pairwise mismatch of either connected components or path-components.

Finally observe that (1.19) shows that compactness is not a homotopy invariant property. On the other hand compactness does have some useful interactions with homotopy.

Proposition 1.8 *The infinite product $\prod_{\mathbb{N}} S^1$ is not homotopy equivalent to a CW complex.*

Proof The key point is that every compact subset of a CW complex is contained in a finite subcomplex, while $\prod_{\mathbb{N}} S^1$ is compact in the product topology. Thus if we assume that X is a CW complex and $\prod_{\mathbb{N}} S^1 \to X \to \prod_{\mathbb{N}} S^1$ is a pair of maps whose composite is homotopic to the identity, then it must be that the first map factors through some finite subcomplex $K \subseteq X$. Restriction therefore gives us maps $\prod_{\mathbb{N}} S^1 \xrightarrow{f} K \xrightarrow{g} \prod_{\mathbb{N}} S^1$ which satisfy $gf \simeq id$.

If H_* denotes singular homology, then using $id_* = (gf)_* = g_* f_*$ we get that the composite

$$H_*(\textstyle\prod_{\mathbb{N}} S^1) \xrightarrow{f_*} H_*K \xrightarrow{g_*} H_*(\textstyle\prod_{\mathbb{N}} S^1) \tag{1.20}$$

is equal to the identity. This shows that $H_*(\prod_{\mathbb{N}} S^1)$ must vanish in sufficiently large degrees. This is true for the reason that all the homology groups of the finite complex K do. On the other hand, including in the first n factors of the product as in the following diagram we see that $\prod_{\mathbb{N}} S^1$ has nonvanishing homology groups in arbitrarily high dimensions

$$\begin{tikzcd} \prod_{i=1}^n S^1 \arrow[r, hook, "in"] \arrow[dr, equal] & \prod_{\mathbb{N}} S^1 \arrow[d, "pr"] \\ & \prod_{i=1}^n S^1 \end{tikzcd} \qquad \begin{tikzcd} H_n(\prod_{i=1}^n S^1) \arrow[r] \arrow[dr, equal] & H_n(\prod_{\mathbb{N}} S^1) \arrow[d, "pr_*"] \\ & H_n(\prod_{i=1}^n S^1) \cong \mathbb{Z}. \end{tikzcd} \tag{1.21}$$

Thus we have a contradiction, and the complex X cannot possibly exist. ∎

A useful notion that is weaker than homotopy equivalence is the following.

Definition 5 *Assume given maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq id_X$. In this case we say that g is a* ***left homotopy inverse*** *to f, and that f is a* ***right homotopy inverse*** *to g.* $\square$

Thus $f : X \to Y$ is a homotopy equivalence when there is a map $g : Y \to X$ which is both a left and right homotopy inverse to it. In fact homotopy inverses have many of the good properties that strict categorical inverses do.

Proposition 1.9 *If $f : X \to Y$ has both a left homotopy inverse g and a right homotopy inverse h, then $g \simeq h$. In particular f is a homotopy equivalence.*

Proof Exercise. ▮

It is also common for authors to call a left (homotopy) inverse a **retraction**, and a right (homotopy) inverse a **section**. Unfortunately it can sometimes be unclear in what sense these terms are meant - whether in a strict or in a homotopy-theoretic sense. This is our reason for introducing the previous terminology.

Example 1.5 A subspace $A \subseteq X$ is said to be a **weak deformation retract** if its inclusion is a homotopy equivalence. i.e. if there is a map $r : X \to A$ and homotopies $F : ir \simeq id_X$ and $G : ri \simeq id_A$. In the case that $ri = id_A$ strictly, then we say that A is a **deformation retract** of X. If furthermore the homotopy F can be chosen to satisfy $F_t|_A = id_A$, $\forall t \in I$, then we say that A is a **strong deformation retract** of X.

Here are some examples:

1. The inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n \setminus 0$ is a strong deformation retract. The retraction is given by $\mathbb{R}^n \setminus 0 \ni x \mapsto \frac{x}{|x|} \in S^{n-1}$, and the required homotopy is $F(x,t) = (1-t)x + t\frac{x}{|x|}$.

2. The subspace $S^{n-1} \times I \cup (D^n \times 0)$ deformation retracts off of $D^n \times I$ by the map

$$r(x,s) = \begin{cases} (x/|x|, 2-(2-s)/|x|) & |x| \geq 1 - \frac{s}{2} \\ (2x/(2-s), 0) & |x| \leq 1 - \frac{s}{2}. \end{cases} \tag{1.22}$$

   The deforming homotopy is given by following the straight line in $D^n \times I$ between (x,s) and $r(x,s)$. Under the standard homeomorphism $D^n \times I \cong I^n \times I$ this deformation retraction becomes one of the subspace $(\partial I^n \times I) \cup (I^n \times 0) \subseteq I^{n+1}$. We'll see when we study *cofibrations* that this example is actually quite useful!

3. The subspace $C \subseteq \mathbb{R}^2$ which is the union

$$C = (I \times \{0\}) \cup (\{0\} \times I) \cup \bigcup_{n \in \mathbb{N}} \{1/n\} \times I \tag{1.23}$$

   is called the **comb space** and is contractible. Actually this space has some strange properties which will make it an interesting counterexample in future.

   A contraction of C is obtained by gluing together the homotopies F, G below. The first is

$$F_t(x,y) = (x, (1-t)y), \qquad (x,y) \in C, t \in I \tag{1.24}$$

which deforms C onto the subspace $I \times \{0\}$. Then

$$G_t(x,0) = ((1-t)x, 0), \qquad (x,0) \in I \times \{0\}, t \in I \tag{1.25}$$

contracts what remains to the point $(0,0)$.

The inclusion $\{0\} \times I \subseteq C$ of the end interval into the comb space C is a homotopy equivalence and a deformation retract. However we will see later that it is *not* a strong deformation retract. Similarly, the inclusion $C \subseteq I^2$ is a homotopy equivalence and a weak deformation retract, but *not* a deformation retract. $\square$

Proposition 1.10 *Let $i : A \overset{\subseteq}{\to} X$ be a subspace inclusion. Then A is a deformation retraction of X if and only if both of the following two conditions are satisfied;*

1. *Each continuous map $A \to Z$ into a space Z is extendable over X.*
2. *If $f, g : X \to Z$ are maps into a space Z and there is a homotopy $f|_A \simeq g|_A : A \to Z$, then there is a homotopy $f \simeq g : X \to Z$.*

Proof Exercise. What do we change if we replace deformation retract with weak/strong deformation retact? ∎

So far we have mainly focused on examples of contractible spaces. Here is a result involving the circle $S^1 \subseteq \mathbb{C}$ and the exponential map

$$\exp : \mathbb{R} \to S^1, \qquad t \mapsto \exp(it) \tag{1.26}$$

which will take us in the opposite direction. Anyone who has some experience with covering spaces will know a much stronger result, but it is the neat proof of the following proposition which is amusing. To my knowledge it is due to S. Eilenberg.

Proposition 1.11 *Let X be a compact metric space and $f : X \to S^1$ a map. Then f is null-homotopic if and only if there exists a map $\varphi : X \to \mathbb{R}$ such that $f = \exp \circ \varphi$.*

Proof If $f = \exp \circ \varphi$, then it is null-homotopic, since $\varphi \simeq *$. To see the converse consider the following motivation. Assume there is a map $\varphi_0 : X \to \mathbb{R}$ such that $|f(x) - \exp(i\varphi_0(x))| < 2$ for all $x \in X$. Note that this implies that $\frac{f(x)}{\exp(i\varphi_0(x))} \neq -1$ for all $x \in X$. Thus if we define $\varphi(x)$ to be the length of the oriented arc in S^1 which goes between 1 and $\frac{f(x)}{\exp(i\varphi_0(x))}$, and which does not pass through -1, then $x \mapsto \varphi(x)$ is a well-defined, continuous map $X \to \mathbb{R}$ satisfying $\frac{f(x)}{\exp(i\varphi_0(x))} = \exp(i\varphi(x))$ for each $x \in X$. In particular

$$f(x) = \exp(i\varphi_0) \cdot \exp(i\varphi(x)) = \exp(i(\varphi_0 + \varphi)(x)), \qquad \forall x \in X. \tag{1.27}$$

Thus to show that f factors through the exponential map, it will be sufficient to find a φ_0 as above.

So, to complete the proof let a null-homotopy $F : * \simeq f$ be given. Since X is compact, F is uniformly continuous and we can find $\delta > 0$ such that $|F_s(x) - F_t(x)| < 2$, $\forall x \in X$, whenever $|s - t| < \delta$. Choose numbers $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $t_{i+1} - t_i < \delta$ for each $i = 0, \ldots, n-1$. Then $F_{t_0} = \exp \circ \varphi_0$, where $\varphi_0 : X \to \mathbb{R}$ is constant at 0. Since $|F_{t_0}(x) - F_{t_1}(x)| < 2$ for all $x \in X$, it follows from the above that we can find $\varphi_1 : X \to \mathbb{R}$ with $F_{t_1} = \exp \circ \varphi_1$. Continuing by induction we obtain $\varphi = \varphi_n : X \to \mathbb{R}$ with $f = \exp \circ \varphi$. ∎

Corollary 1.12 *The circle S^1 is not contractible.*

Proof We show that any self-homeomorphism of S^1 must be essential. Thus let $\alpha : S^1 \xrightarrow{\cong} S^1$ be any homeomorphism and assume that it is nullhomotopic. Then according to Proposition 1.11 it has a lift $\widehat{\alpha} : S^1 \to \mathbb{R}$ through exp. The map $\widehat{\alpha}$ must be injective, so it follows by compactness and connectedness that it maps S^1 bijectively onto a closed subinterval of $\mathbb{R}$. Since $\mathbb{R}$ is Hausdorff, $\widehat{\alpha}$ must be a homeomorphism onto its image. But this is a contradiction, since S^1 is not homeomorphic to an interval: removing an interior point from an interval yields a disconnected space. ∎

As we have pointed out, this is not a particularly strong result and we will soon surpass it. We include it for two reasons. Firstly because its proof has a flavour different from many of the other proofs we will subsequently give in these notes. We will tend to favour a slight more abstract and conceptual approach than the very hands-on approach above. The statement and its proof serve as a reminder that basic techniques can often be very effective.

A second reason that we have chosen to highlight this statement is simply for the fact that we have not quite yet demonstrated that homotopy theory is actually interesting. Now we know: not every path-connected space is homotopy equivalent to a point. We did remark in Corollary 1.6 that path connectedness is necessary for contractibility, and we can consider Corollary 1.12 to be the first extension of this.

Below we give two more basic applications of our theory to some slightly more interesting geometric problems. As a warm up we'll introduce the following.

Definition 6 *A space X is said to be* ***locally contractible at*** *$x \in X$ if for each neighbourhood U of x there exists a neighbourhood $V \subseteq U$ of x such that the inclusion $V \hookrightarrow U$ is inessential. The space X is said to be* ***locally contractible*** *if it is locally contractible at each of its points.* □

Competing definitions for local contractibility appear in the literature. Manifolds are locally contractible, as are CW complexes [2] pg.28. Notice how the definition functions: although the comb space is contractible, it is not locally contractible (consider the points on $\{0\} \times I$). In particular local contractibility is not an invariant of homotopy type.

Example 1.6 For $n \geq 0$ let $\mathcal{H}_\partial(D^n)$ be the group of all homeomorphisms $\phi : D^n \xrightarrow{\cong} D^n$ which satisfy $\phi|_{S^{n-1}} = id_{S^{n-1}}$. We give $\mathcal{H}_\partial(D^n)$ the subspace topology inherited from $C(D^n, D^n)$ with the compact-open topology[3], and this makes $\mathcal{H}_\partial(D^n)$ into a topological group. Each $f \in \mathcal{H}_\partial(D^n)$ has a canonical extension over all of $\mathbb{R}^n$ by letting it act as the

[3] For a space X let $\mathcal{H}(X) \subseteq C(X, X)$ be the set of all self-homeomorphisms $X \xrightarrow{\cong} X$ in the compact-open topology. In general this is not a topological group. If X is locally compact, then composition is continuous and $\mathcal{H}(X)$ becomes a topological monoid under composition. Continuity of inversion, however, is still problematic. If we assume that X is compact Hausdorff, then we do get a continuous inversion and $\mathcal{H}(X)$ becomes a topological group (note that if $f \in \langle K, U\rangle$, then $f^{-1} \in \langle X \setminus U, X \setminus K\rangle$). Also inversion is continuous in the case that X is locally connected, locally compact Hausdorff, in which case $\mathcal{H}(X)$ is a topological group. In particular $\mathcal{H}(X)$ is a topological group if *i)* X is a finite CW complex, or *ii)* X is a topological manifold. A (topological) **isotopy** of a space X is a homotopy $h : X \times I \to X$ such that $h_t : X \xrightarrow{\cong} X$ is a homeomorphism for each $t \in I$. If X is locally compact Hausdorff, then isotopies of X are exactly paths $I \to \mathcal{H}(X)$.

identity outside of D^n. With this in mind we define $\lambda : \mathcal{H}_\partial(D^n) \times \mathcal{H}_\partial(D^n) \times I \to \mathcal{H}_\partial(D^n)$ by putting

$$\lambda(f, g; t)(x) = \begin{cases} f(x) & t = 0 \\ f(tf^{-1}(g(x/t))) & t \in (0, 1]. \end{cases} \tag{1.28}$$

Then λ is a continuous map which for all $f, g \in \mathcal{H}_\partial(D^2)$ satisfies

- $\lambda(f, g; 0) = f$
- $\lambda(f, g; 1) = g$
- $\lambda(f, f; t) = f$ for all $t \in I$.

In particular

$$F : (f, t) \mapsto \lambda(f, id_{D^n}; t) \tag{1.29}$$

gives a contraction of $\mathcal{H}_\partial(D^n)$.

Notice, however, that λ does much more than merely contract $\mathcal{H}_\partial(D^n)$: it is a homotopy $pr_1 \simeq pr_2$ which is fixed on the diagonal. To put this observation to use fix $f \in \mathcal{H}_D(D^n)$ and let $U \subseteq \mathcal{H}_D(D^n)$ be an open neighbourhood of f. By the third property listed above we can use the continuity of λ and the tube lemma to find an open set $V \subseteq U$ containing f such that $\lambda(V \times \{f\} \times I) \subseteq U$. By restriction λ now defines a contraction of V in U, which shows that $\mathcal{H}_\partial(D^n)$ is locally contractible at f. Of course f was arbitrary and in this paragraph we have shown the following.

Proposition 1.13 *The space $\mathcal{H}_\partial(D^n)$ is both contractible and locally contractible.* ∎

There is a lot of geometric intuition behind the definition of α, which can be traced back to Alexander's 1923 paper [1]. It is known as *Alexander's trick.* □

Example 1.7 The inclusion[4]

$$O_n \hookrightarrow Gl_n(\mathbb{R}) \tag{1.30}$$

is a homotopy equivalence. We construct the inverse as follows. If $A \in Gl_n(\mathbb{R})$, then $A^T A$ is positive definite and the equation $P_A^2 = A^T A$ uniquely defines a symmetric, positive definite matrix P_A[5]. Note that P_A is invertible, so from the fact that P_A is symmetric we see that AP_A^{-1} is orthogonal.

Writing $\mathcal{P}_n$ for the space of positive definite real $n \times n$ matrices (topologised as a subspace of $M_n(\mathbb{R})$) consider the assignment

$$\begin{aligned} Gl_n(\mathbb{R}) &\to O_n \times \mathcal{P}_n \\ A &\mapsto (AP_A^{-1}, P_A). \end{aligned} \tag{1.31}$$

[4]Here, $M_n(\mathbb{K})$ denotes the set of all $n \times n$ matrices with entries in a given field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The subspace $Gl_n(\mathbb{K}) \subseteq M_n(\mathbb{K})$ denotes the topological group of all $n \times n$ matrices with non-vanishing determinant. It is called the **general linear group** over $\mathbb{K}$ of degree n, and is a non-compact Lie group. i.e. it has the structure of a smooth manifold and the operations of multiplication and inversion are smooth maps. When $\mathbb{K} = \mathbb{R}$, the subgroup $O_n \subseteq Gl_n(\mathbb{R})$ consists of those matrices A satisfying $A^T A = AA^T = I$, and is called the **orthogonal group** of degree n. It is a compact, closed Lie subgroup of $Gl_n(\mathbb{R})$, and is precisely the group of orthogonal transformations of $\mathbb{R}^n$ with its Euclidean inner product. For this course you will only need to understand these objects as groups of matrices.

[5]A symmetric matrix $P \in M_n(\mathbb{R})$ is said to be **positive definite** if $v^T P v > 0$ for all $v \neq 0 \in \mathbb{R}^n$.

Each matrix P_A may be computed as a power series, so this map is continuous. Since we can recover A as $A = (AP_A^{-1})P_A$, this map is in fact a homeomorphism.

Now $\mathcal{P}_n$ is $*$-convex around I and thus contractible. This means that the projection $O_n \times \mathcal{P}_n \to O_n$ is a homotopy equivalence, whose inverse is the inclusion of O_n at any point ($I \in \mathcal{P}_n$ is a canonical basepoint). Composing this projection with (1.31) we get a map $Gl_n(\mathbb{R}) \to O_n$, $A \mapsto AP_A^{-1}$, which is the homotopy inverse to (1.30). Note that we have proved more than originally claimed. If $A \in Gl_n(\mathbb{R})$ is orthogonal, then $P_A = I$. Thus the above map exhibits O_n as a strong deformation retraction of $Gl_n(\mathbb{R})$.

Similar considerations give rise to a complex version of the statement: the inclusion of the unitary group

$$U_n \hookrightarrow Gl_n(\mathbb{C}) \tag{1.32}$$

is a homotopy equivalence and strong deformation retract. □

Finally we will end these notes by introducing the notion of a 'homotopy of homotopies' which we shall need in future. It is slightly subtle, but it will turn out to be important to us to understand not only when two maps are homotopic, but also the exact manner in which they are. It is too much to reasonably ask to understand all the possible homotopies between a pair of maps, but the notion we are about to introduce captures just what we will need.

Definition 7 *Let $G, H : f \simeq g$ be two homotopies of maps $X \to Y$. We say that G is* ***track homotopic*** *to H, written $G \sim H$, if there is a continuous map $\Psi : X \times I \times I \to Y$ satisfying*

1. $\Psi(x, t, 0) = G(x, t)$
2. $\Psi(x, t, 1) = H(x, t)$
3. $\Psi(x, 0, s) = f(x)$ *for all* $x \in X$, $s \in I$
4. $\Psi(x, 1, s) = g(x)$ *for all* $x \in X$, $s \in I$. □

Track homotopy leads to a groupoid structure on the set of continuous maps $X \to Y$, and is pleasantly well behaved in general. We will not formalise this, but see tom Dieck [3] §2.9 for further details.

Proposition 1.14 *If $F : f \simeq g : X \to Y$ is a homotopy, then*

$$f + F \sim F \sim F + g \qquad \text{and} \qquad F - F \sim f. \tag{1.33}$$

If $G : g \simeq h$ and $H : h \simeq k$ are two further homotopies of maps $X \to Y$, then

$$(F + G) + H \sim F + (G + H). \tag{1.34}$$

In the case that there are track homotopies $F \sim F'$ and $G \sim G'$, then

$$F + G \sim F' + G'. \quad \blacksquare \tag{1.35}$$

The proof of this is straightforwards and omitted. It can be much streamlined by use of the following lemma.

Lemma 1.15 *Let H be a homotopy of maps $X \to Y$. If $\varphi : I \to I$ is a map satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$, then there is a track homotopy $H \sim H(1_X \times \varphi)$.*

Proof Define $\psi : X \times I \times I \to Y$ by

$$\psi(x, s, t) = H(x, (1-t)s + t\varphi(s)), \qquad x \in X, s, t \in I. \quad \blacksquare \tag{1.36}$$

Note that the horizontal composition also works well with track homotopy, and the following equations hold whenever they make sense.

- $(F \star G) \star H = F \star (G \star H)$ (strictly).
- $f \star G \star h = fGh$.
- $F \star G \sim F' \star G'$ whenever $F \sim F'$ and $G \sim G'$.
- $F \star (G + H) \sim fG + (F \star H) \sim (F \star G) + gH$, where $F : f \simeq g$ (this is just the interchange law 1.4).
- $(F + G) \star H \sim (F \star H) + Gk \sim Fh + (G \star H)$, where $H : h \simeq k$ (this is again the interchange law).

Example 1.8

1. If V is a vector space, then any two maps $f, g : X \to V$ are homotopic. If $F, G : f \simeq g$ are homotopies, then F, G are track homotopic

$$\psi_s(x, t) = (1-s)F(x, t) + sG(x, t). \tag{1.37}$$

   Note that again this homotopy is canonical. On the other hand, if we replace V with an affine space A, then the two maps are homotopic, and the two homotopies are track homotopic, but nothing in this case is canonical.

2. A point of X is a map $x_0 : * \to X$ from the 1-point space. A path $\gamma : I \to X$ from x_0 to x_1 is then a homotopy $\gamma : x_0 \simeq x_1$. A path homotopy $\gamma \simeq \gamma'$ is thus the same thing as a track homotopy. The notions of concatenation of paths and of addition of homotopies coincide. Attempting to use these observations as definitions would be clumsy, but would have the benefits of immediately recognising that topological properties like path connectedness are really homotopy-theoretic concepts. The point of noticing this connection that it brings some intuition to the associative-up-to-homotopy properties 1.14 that track addition enjoys. □

1.1 Exercises

Maps into Spheres

1. Show that any non-surjective map $f : X \to S^n$ is inessential.

2. Show that if $f, g : X \to S^n$ are maps satisfying $f(x) \neq -g(x)$, $\forall x \in X$, then $f \simeq g$. Show that the implication is not reversible by finding an integer n, a space X, and maps $f, g : X \to S^n$ such that $f \simeq g$, but $f(x) = -g(x)$, $\forall x \in X$.

3. Show that the unreduced cone on S^n is homeomorphic to the disc D^{n+1}. Use this observation to construct two 'preferred' null-homotopies of the equatorial inclusion $S^n \hookrightarrow S^{n+1}$. Subsequently use the two preferred null-homotopies to construct four different self-maps $S^{n+1} \to S^{n+1}$. Show that there are at least two distinct homotopy classes amongst these four self-maps. You may take for granted that S^{n+1} is not contractible. (There are actually three distinct homotopy classes here. Feel free to attempt a proof of this fact.)

Constructing Homotopies

1. Let X be a space and let $\{Y_i\}_{i \in \mathcal{I}}$ be a family of spaces. Assume given a family of homotopies $F^{(i)} : f_i \simeq g_i : X \to Y_i$. By writing the homotopies in adjoint form $F^{(i)} : X \to Y_i^I$, show that the two maps $(f_i)_{\mathcal{I}}, (g_i)_{\mathcal{I}} : X \to \prod_{\mathcal{I}} Y_i$ are homotopic.

2. Suppose given a family of spaces $\{X_i\}_{i \in \mathcal{I}}$ and a family $F^{(i)} : f_i \simeq g_i : X_i \to Y$ of homotopies into a space Y. Construct a homotopy between the coproduct maps $(f_i)_{\mathcal{I}}, (g_i)_{\mathcal{I}} : \coprod_{\mathcal{I}} X_i \to Y$.

3. Let $f, g : X \to Y$ be a pair of maps between spaces X, Y. Assume $X = U \cup V$ is a union of open (closed) subsets $U, V \subseteq X$ and there is given a pair of homotopies $F^U : f|_U \simeq g|_U : U \times I \to Y$ and $F^V : f|_V \simeq g|_V : V \times I \to Y$ such that $F^U(x,t) = F^V(x,t)$ for all $x \in U \cap V$ and $t \in I$. Then there is a homotopy $F : f \simeq g : X \times I \to Y$. Prove this in the following two ways; i) by using the fact that $(-) \times I$ preserves pushouts, since I is locally compact, ii) by taking adjoints of the homotopies F^U, F^V. Form an objection to these two methods being described as 'different'.

Deformation Retracts

1. Assume that $A \subseteq \mathring{D}^n$ is $*$-convex about the origin. Show that $S^{n-1} \subseteq \mathbb{R}^n \setminus A$ is a deformation retract.

2. Let $A \subseteq B \subseteq X$. Assume that both $A \subseteq B$ and $B \subseteq X$ are deformation retracts. Show that $A \subseteq X$ is a deformation retract. Show that $A \subseteq X$ is a strong deformation retract if both the deformation retracts $A \subseteq B$ and $B \subseteq X$ are strong.

3. Show that the pushout of a strong deformation retract is a strong deformation retract. Is the statement true if we replace strong deformation retract with deformation retract? With weak deformation retract?

4. Show that a strict retract of a Hausdorff space X is necessarily closed in X. Use this to find an example of a weak deformation retraction which is not a deformation retraction.

Local Path Connectedness and Local Contractibility

1. A nonempty space X is path connected if for any pair of points $x_0, x_1 \in X$, there is a path $l : I \to X$ with $l(0) = x_0$ and $l(1) = x_1$. Formulate an equivalant definition for this in terms of homotopy class of maps $* \to X$ and use it to conclude that 'path connectedness' is an invariant of homotopy type.

2. A space X is said to be **locally path connected** if the path components of each open subset $U \subseteq X$ are themselves open. Give an example to show that 'local path connectedness' is *not* an invariant of homotopy type.

3. A space X is said to be **locally contractible at** $x \in X$ if for each neighbourhood U of x there exists a neighbourhood $V \subseteq U$ of x such that the inclusion $V \xrightarrow{\subseteq} U$ is inessential. X is said to be **locally contractible** if it is locally contractible at each of its points. Show that a locally contractible space is locally path connected. Give an example to show that local contractibility is *not* an invariant of homotopy type.

Semilocal Contractibility

1. A space X is said to be **semilocally contractible** if each point $x \in X$ has a neighbourhood $U \subseteq X$ whose inclusion into X is inessential. Show that if X is semilocally contractible, then so too is any homotopy retract of X. Conclude that semilocal contractibility is an invariant of homotopy type.

2. Show that if X is locally contractible and $Y \simeq X$, then Y is semilocally contractible.

3. Show that the path components of a semilocally contractible space are open. Conclude that X is semilocally contractible if and only if its path components are open and are themselves semilocally contractible.

4. Show that if X is any space, then the unreduced suspension $\widetilde{\Sigma}X$ is semilocally contractible. If X is a pointed space, then is the same necessarily true for the reduced suspension ΣX?[6]

5. Show that a finite product of semilocally contractible spaces is itself semilocally contractible. Is the same true for an infinite product of semilocally contractible spaces?

6. Let A be semilocally contractible and $\varphi : S^{n-1} \to A$ a map. Show that the adjunction space $A \cup_\varphi e^n$ is semilocally contractible.

References

[1] J. Alexander, *On the Deformation of an n-Cell*, Proc. Nat. Acad. Sci. USA., **12**, (1923), 406-407.

[6] The *unreduced suspension* of an unpointed space is defined in 'Exercise Sheet 2: Introduction to Lusternik-Schnirelmann Category'. The *reduced suspension* of a pointed space is defined in 'Pointed Homotopy'.

[2] R. Fritsch, R. Piccinini, *Cellular Structures in Topology*, Cambridge University Press, (1990).

[3] T. tom Dieck, *Algebraic Topology*, European Mathematical Society, (2008).